

Remarks on the realization of the Atiyah-Singer index theorem in lattice gauge theory*

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We discuss the interplay between topologically non-trivial gauge field configurations and the spectrum of the Wilson-Dirac operator in lattice gauge theory. Our analysis is based on analytic arguments and numerical results from a lattice simulation of QED₂.

1. MOTIVATION

It has been known [1] for more than two decades that topologically non-trivial configurations play an essential role for a proper understanding of QCD. However, it is not straightforward to go beyond a semiclassical analysis when implementing topological ideas in the continuum path integral. There are at least two reasons: Firstly the continuum path integral (if it can be constructed at all) has support only on configurations without the necessary smoothness condition for a definition of the topological charge - in general the gauge fields contributing to the continuum path integral are simply too rough to be classified with respect to their topological charge. Secondly one would also like to use the Atiyah-Singer index theorem [2] in the fully quantized theory. It relates the topological charge of a differentiable background configuration to the zero modes of the Dirac operator. Again the lack of smoothness of the gauge fields contributing to the path integral does not allow the application of the index theorem beyond a semiclassical analysis.

On the lattice the situation is different. There are several definitions of the topological charge on the lattice which allow to classify (almost) all gauge field configurations contributing in the continuum limit. In contrary to the continuum a

decomposition of the path integral into topological sectors is possible. Concerning the Atiyah-Singer index theorem, the situation is more difficult. There is no analytic result for an index theorem on the lattice, but it is generally believed that an equivalent result should be manifest on the lattice in a probabilistic way.

Although several aspects of the interplay between topologically non-trivial configurations and the spectrum of the Dirac operator on the lattice have been analyzed [3], due to the immense computational cost no complete study in fully quantized QCD₄ has been accomplished. The numerical data which we present in this contribution come from a study [4] in lattice QED₂. This model has similar structural features as QCD₄ (topologically non-trivial configurations, existence of the anomaly, U(1)-problem etc.) but is computationally much less demanding. This allows for a proper understanding of the role of topology in that simple model, and indicates what to expect for a lattice study of QCD₄.

2. SYMMETRIES OF THE WILSON-DIRAC OPERATOR

Before numerically analyzing the spectrum of the Wilson-Dirac operator M it is helpful to reassess the symmetry properties of the operator (fermion matrix) and thus of the spectrum. We denote the fermion matrix as $M = 1 - \kappa Q$, where κ is the hopping parameter and Q the hopping

*Supported by Fonds zur Förderung der Wissenschaftlichen Forschung in Österreich, Projects P11502-PHY and J01185-PHY.

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matrix

$$Q(x, y) = \sum_{\nu=\pm 1}^{\pm D} (1 + \gamma_\nu) U_\nu(x - \hat{\nu}) \delta_{x-\hat{\nu}, y}, \quad (1)$$

where we use the notation $\gamma_{-\nu} = -\gamma_\nu$ and $U_{-\nu}(x + \hat{\nu}) = U_\nu(x)^\dagger$. D is the number of dimensions (2 or 4) and $U_\nu(x)$ are elements of the gauge group (U(1) for $D = 2$ and SU(N) for $D = 4$). We work on a finite $L^{D-1} \times T$ lattice with mixed periodic boundary conditions. The existence of the following similarity transformations is well known [3]

$$\Gamma_5 Q \Gamma_5 = Q^\dagger, \quad \Xi Q \Xi = -Q, \quad (2)$$

where the latter only holds for even L and T . The matrices $\Gamma_5 = \gamma_5 \delta_{x,y}$ and $\Xi = (-1)^{x_1 + \dots + x_D} \delta_{x,y}$ are both unitary and hermitian. This implies that the spectrum of Q is symmetric with respect to reflection on both the real and the imaginary axis. Thus the eigenvalues of the hopping matrix on a lattice with even L and T come in complex quadruples or in real pairs.

The special role of the *real* eigenvalues λ of the hopping matrix (and thus the full fermion matrix $M = 1 - \kappa Q$) can be appreciated only by understanding the chiral properties of their eigenvectors. Denote by v_λ, v_μ the eigenvectors of Q with eigenvalues λ, μ . Using the symmetries (2) it can be shown [4] that

$$v_\mu^\dagger \Gamma_5 v_\lambda \neq 0, \quad (3)$$

only for $\mu = \bar{\lambda}$. In particular for the diagonal elements (this result can already be found in [5])

$$v_\lambda^\dagger \Gamma_5 v_\lambda \neq 0, \quad (4)$$

only for *real* eigenvalues λ . This result has to be compared to the continuum result (ψ denotes some eigenstate of the continuum Dirac operator)

$$(\psi, \gamma_5 \psi) \neq 0, \quad (5)$$

only if ψ is a *zero mode*. This indicates that only the eigenvectors of the fermion matrix corresponding to real eigenvalues can play the role of the zero modes of the continuum.

At this point we would like to remark that for an analysis of the realizations of index theorems

on the lattice the Wilson form of the lattice Dirac operator is superior to the staggered version. The latter is antihermitian and thus has only purely complex eigenvalues lacking a straightforward criterion for identifying zero modes. Furthermore it is also not clear how to relate the (real) eigenvalues of the hermitian operator $\Gamma_5 Q$ to the spectrum of Q with its more sophisticated features.

3. NUMERICAL RESULTS IN QED₂

Having established the real eigenvalues of the fermion matrix as the trace of gauge field configurations with non-trivial topology one can simplify a numerical analysis by concentrating on the real eigenvalues alone. Here we discuss results from a numerical simulation of QED₂ on the lattice [4]. In this model the evaluation of the geometric definition [6] of the topological charge $\nu[U]$ is straightforward, giving

$$\nu[U] = \frac{1}{2\pi} \sum_{x \in \Lambda} \theta_P(x) \in \mathbb{Z}. \quad (6)$$

The plaquette angle $\theta_P(x)$ is introduced as $\theta_P(x) = \text{Im} \ln U_P(x)$ where U_P are the ordered products of the link variables U_ν around the plaquette. In order to understand an eventual realization of the index theorem on the lattice one would like to establish a relation between the number of real eigenvalues (compare the discussion above) of the fermion matrix and the functional $\nu[U]$.

After inspecting spectra for several configurations one can conjecture the rule

$$\# \text{ of real eigenvalues} = 4 |\nu|. \quad (7)$$

It has to be remarked, that this is not a result in a mathematically strict sense, since isolated gauge field configurations which violate (7) can be constructed. However, it can be tested numerically whether (7) holds for a considerable portion of the gauge field configurations of the lattice path integral. In order to analyze this question we performed a simulation of lattice QED₂ with two flavors of dynamical fermions at several values of β (inverse coupling squared) and L ($T = L$). The value of κ was always chosen close to the critical κ for the respective values of β and L . For

each Monte Carlo configuration the topological charge (6) was evaluated. We define $p(\beta)$ to be the probability of finding (7) correct for a given β (and L).

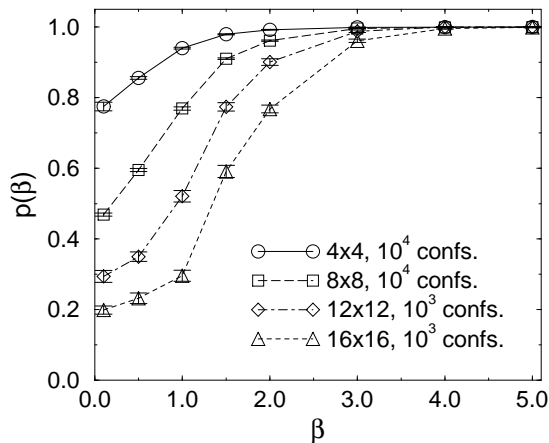


Figure 1. Probability $p(\beta)$ of finding (7) correct as a function of β . We show our results for lattice sizes $L = 4, 8, 12$ and 16 . The symbols are connected to guide the eye.

It is obvious from fig. 1 that already for moderately high values of β the configurations obeying (7) entirely dominate the lattice path integral. This establishes, that with the proper interpretation of the real eigenvalues and their eigenvectors as the trace of configurations with non-trivial topology, the index theorem holds in a probabilistic sense. In particular in the continuum limit $\beta \rightarrow \infty$ (7) holds with probability 1.

At this point we should point out the particular features of QED_2 as opposed to QCD_4 . For QED_2 it is known [7], that in addition to the Atiyah Singer index theorem there holds the so-called *vanishing theorem*, which states that there are either only left-handed or only right-handed zero modes. Thus the result (7) is exactly what one expects from the index and vanishing theorems for smooth continuum configurations (the factor 4 comes from the extra real eigenvalues caused by the doublers). In QCD there is no vanishing theorem. Thus one expects the *difference* of the numbers of left- and right-handed real eigenvalues to be related to the topological charge.

How can one utilize an understanding of the realization of the index theorem on the lattice? In general all *semiclassical* continuum arguments involving the index theorem can be extended to the *fully quantized* lattice model. In [4] it was e.g. demonstrated how the dependence of the pseudoscalar density on the topological charge can be understood using the index theorem. In addition one can use the results for the spectrum to analyze problems that are specific for the lattice approach. In particular in [4] we investigated the behaviour of the sign of the fermion determinant, establishing that it can become negative only for configurations with non-trivial topology.

We believe, that extending the numerical part of our analysis to QCD_4 will give interesting insight on the role of topologically non-trivial gauge field configurations in the fully quantized model.

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